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by

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A PRIORI ERROR BOUNDS FOR PROCUREMENT COMMODITY AGGREGATION IN LOGISTICS PLANNING MODELS

by

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ABSTRACT

A complete logistical planning model of a firm or public system should include activities having to do with the procurement of supplies. Not infrequently, however, procurement aspects are difficult to model because of their relatively complex and evanescent nature. This raises the issue of how to build an overall logistics model in spite of such difficulties. This paper offers some suggestions toward this end which enable the procurement side of a model to be simplified via commodity aggregation in a "controlled" way, that is, in such a manner that the modeler can know and control in advance of solving his model how much loss of accuracy will be incurred for the solutions to the (aggregated) overall model.

I. INTRODUCTION

In this paper the term *procurement* is used in a broad sense that includes materials management of parts and raw materials for a manufacturing firm, the acquisition of goods for subsequent distribution by a wholesale firm, the procurement of supplies and materials by a service organization and similar situations. The essential point is that we are addressing the "initial" rather than the "final" stage of a logistics system. See, for instance, the recent book by D. Bowersox [2] which makes the distinction in terms of material management (supplier-oriented) and physical distribution management (customer-oriented).

Whereas it is the large number of customers and their ordering idiosyncrasies that tend to make the final stage of a logistics system hard to model, it is the large number of suppliers and items and sometimes the constantly changing patterns of procurement that frequently make the initial stage difficult to model. Aggregation of customers on a geographic basis into customer zones and aggregation of delivered products (or services) into product groups are commonly used to simplify the final stage of a logistics planning model. Similar aggregations can be used to simplify the initial stage, but satisfactory simplifications may be more difficult to achieve because of the influence of differential supply costs among suppliers and the greater degree of uniqueness as to which suppliers provide what. These influences seem to call for a relatively greater amount of detail to be preserved in the procurement stage of a planning model. Unfortunately, this could require the preparation of unduly detailed procurement forecasts -- which suppliers will be able to supply what items at what prices in what annual quantities. The difficulties of assembling this data could be out of proportion to the relative importance of procurement as a component of the total logistics planning model. Even worse, it may not be sensible to impose strict model control in the traditional linear programming sense over procurement activities at so great a level of detail.

A reasonable response to these possible difficulties is to take a more flexible attitude toward the modeling of procurement than is customary among devotees to mathematical programming. Namely, look upon the procurement pattern as an aspect of the problem that is partly given objectively and partly under the analyst's control as though it were a policy parameter. View the procurement pattern as something whose influence is as much to be understood as it is to be "optimized".

The aim of this paper is to provide a rigorous framework within which this flexible modeling attitude can be exercised. We are particularly interested in à priori error bounds concerning the accuracy of the full logistics planning model as it is influenced by aggregating procurement items. So far as we are aware, our results along these lines are without precedent.

A companion paper [5] develops similar results in the context of customer aggregation.

II. MODELING STRATEGIES

As a point of departure, consider the following logistics planning model.

Planning Model P

$$\text{(1)}\quad \underset{x,y,z}{\text{minimize}} \sum_{ijk} c_{ijk}\, x_{ijk} + F(y,z)$$

$$\text{(2)}\quad \text{subj. to} \quad \underline{S}_{ij} \leq \sum_{k} x_{ijk} \leq \overline{S}_{ij}\ , \quad \text{all } i\ j$$

$$\text{(3)}\quad \sum_{j} x_{ijk} = \sum_{\ell} D_{i\ell}\, y_{k\ell}\ , \quad \text{all } i\ k$$

$$\text{(4)}\quad \sum_{k} y_{k\ell} = 1, \quad \text{all } \ell$$

$$\text{(5)}\quad x_{ijk} \geq 0\ , \quad \text{all } i\ j\ k$$

$$\text{(6)}\quad y_{k\ell} \geq 0\ , \quad \text{all } k\ell \text{ and } (y,z) \in \Omega\ .$$

The following interpretations will be used:

- i indexes procurement *items* (raw materials, parts, finished goods, etc.)
- j indexes geographical *procurement zones*
- k indexes the *facilities* being supplied
- ℓ indexes *customers*
- x_{ijk} a variable giving the annual amount of item i procured from zone j for facility k
- $y_{k\ell}$ a variable giving the fraction of the annual needs of customer ℓ (for goods or services) satisfied by facility k
- z a vector of additional (possibly logistical) variables
- c_{ijk} unit cost of procurement plus transportation associated with the flow x_{ijk}

$F(y,z)$ the total annual costs associated with (y,z) exclusive of procurement and inbound transportation (typically, facility-related costs plus outbound transportation costs)

$\underline{S}_{ij}(\overline{S}_{ij})$ a lower (upper) limit on the annual amount of item i procured from zone j (partly given and partly at the analyst's discretion)

$D_{i\ell}$ the amount of item i required to satisfy the total annual needs of customer ℓ

Ω a constraint set that must be satisfied by (y,z).

It is understood that a list L_x of allowable triples (i,j,k) is given to reflect which procurement zones can provide which items to which facilities, and a list L_y is given to specify which facilities can serve which customers. All summations and constraint enumerations run only over allowable combinations. For instance, the enumeration in (2) over "ij" runs over the pairs $(\overline{i},\overline{j})$ such that $(\overline{i},\overline{j},k) \in L_x$ for some k.

Constraints (2) control the procurement pattern. An historical procurement pattern (or some other preconceived pattern) can be enforced by taking corresponding $\underline{S}_{ij}$ and $\overline{S}_{ij}$'s to be the same or nearly the same. The latitude for departure from the preconceived pattern increases as $\overline{S}_{ij} - \underline{S}_{ij}$ increases. A necessary condition for feasibility is

(7) $$\sum_j \underline{S}_{ij} \leq \sum_\ell D_{i\ell} \leq \sum_j \overline{S}_{ij} \quad \text{for all } i .$$

The objective function (1) gives the total cost associated with logistical activities. We have already discussed (2). Constraints (3) specify that each facility must receive exactly enough of each item to satisfy the needs of the customers it serves. This requires that the goods or services demanded by each customer can be converted into corresponding requirements for the constituent items (it is immaterial whether the facilities do manufacturing or distribution or service or some combination thereof). Constraints (4) specify that the full needs of each customer must be satisfied. Constraints (5) and (6) impose whatever other requirements on the variables may be needed for system feasibility.

Observe that for fixed y and z, the optimization over x separates into independent subproblems for each i -- each a slight generalization of the classical minimum cost transportation problem.

Because of the complete generality of F and Ω, the model could be set up to determine the least cost facility locations satisfying a desired level of customer service. Normally this would require that F be discontinuous in order to accomodate fixed costs, or some binary z-variables could be introduced to achieve the same effect. The model could also be set up to provide for multiple commodities flowing to customers from the facilities, unique assignment of customers to facilities for certain commodities, and many other problem features. We prefer to leave the model in its general form (1) - (6) because these and many other special cases are thereby treated simultaneously with minimum notational complexity.

The model as stated is actually just a point of departure for the models we actually wish to study. Its chief shortcoming is that it may involve too great a level of detail regarding procurement from the viewpoint of policy and also sheer size. Consider first the policy aspect. Model P places limits on the procurement pattern (via (2)) on an item-by-item basis. Except for items of major importance, this seems like an excessive degree of control and may not even be meaningful in situations where suppliers are changed frequently on the basis of current price and availability. It would make more sense when there are many items of small importance to aggregate some of the constraints in (2). Suppose this is done for some subset I of items. The result is

Planning Model P_I

The same as planning model P, except that (2) is replaced by

(2.1) $\underline{S}_{ij} \leq \sum_k x_{ijk} \leq \overline{S}_{ij}$, all i j with $i \notin I$

(2.2) $\underline{S}_{I,j} \leq \sum_{\substack{i\epsilon I \\ k}} x_{ijk} \leq \overline{S}_{I,j}$, all j , where

(8) $\underline{S}_{I,j} \triangleq \sum_{i\epsilon I} \underline{S}_{ij}$ and $\overline{S}_{I,j} \triangleq \sum_{i\epsilon I} \overline{S}_{ij}$.

This version seems more reasonable from a policy standpoint in that the procurement pattern for items I is now stipulated on an aggregate basis. The numbers $\underline{S}_{I,j}$ and $\overline{S}_{I,j}$ would be interpreted rather freely since their formal constituents $\underline{S}_{ij}$ and $\overline{S}_{ij}$ might be poorly known or perhaps even ill-defined.

There is, of course, a natural generalization of P_I that aggregates the procurement pattern constraints for several subsets of items. The analysis of this generalization is a simple extension of the results to be obtained for P_I (see the Remark in Appendix 1).

Model P_I is better from a policy standpoint but it still may be too large. The number of variables is unchanged, although the number of type

(2) constraints has diminished. Moreover, a possible new difficulty arises in that the mathematical structure of P_I is more complex than that of P. This is due to the fact that aggregating the type (2) constraints over $i \in I$ has the effect of coupling together what previously was a collection of independent transportation-like subproblems in the x-variables when y and z are fixed. The new coupling tends to diminish the computational effectiveness of solution methods that exploit the natural separation into subproblems when y and z are held fixed temporarily (e.g., methods based on Benders decomposition [4]). The nice structure of P could be restored, and the size of P_I much reduced, by completing the aggregation with respect to items I begun in the passage from P to P_I. This involves replacement of the variables x_{ijk} with $i \in I$ by aggregate variables ξ_{jk}, say, so that the following single transportation-like subproblem replaces the coupled subproblems of P_I for fixed y:

$$\underset{\xi \geq 0}{\text{Minimize}} \quad \sum_{jk} b_{jk} \xi_{jk}$$

subj. to

$$\text{(2.2A)} \qquad \underline{S}_{I,j} \leq \sum_{k} \xi_{jk} \leq \overline{S}_{I,j} \quad , \text{ all } j$$

$$\text{(3.1)} \qquad \sum_{j} \xi_{jk} = \sum_{\substack{i \in I \\ \ell}} D_{i\ell}\, y_{k\ell}, \text{ all } k \ ,$$

where the b_{jk}'s are plausible surrogates for the c_{ijk}'s over $i \in I$. Variable ξ_{jk} is interpreted as a surrogate for $\sum_{i \in I} x_{ijk}$, and (3.1) is interpreted as requiring facility k to receive enough of the items in I to meet its needs in the aggregate.

This further aggregation of P_I leads to

Planning Model $\tilde{P}_{I,b}$

(9) $$\underset{x,y,z,\xi}{\text{Minimize}} \quad \sum_{i \notin I} \sum_{jk} c_{ijk}\, x_{ijk} + \Sigma_{jk}\, b_{jk}\xi_{jk} + F(y,z) + L(y;b)$$

subject to

(2.1) $$\underline{S}_{ij} \le \sum_k x_{ijk} \le \overline{S}_{ij}\ , \quad \text{all } i\ j \text{ with } i \notin I$$

(2.2A) $$\underline{S}_{I,j} \le \sum_k \xi_{jk} \le \overline{S}_{I,j}\ , \quad \text{all } j$$

(3.1) $$\sum_j \xi_{jk} = \sum_\ell D_{I,\ell}\, y_{k\ell}\ , \quad \text{all } k$$

(3.2) $$\sum_j x_{ijk} = \sum_\ell D_{i\ell}\, y_{k\ell}\ , \quad \text{all } i\ k \text{ with } i \notin I$$

(4) $$\sum_k y_{k\ell} = 1\ , \quad \text{all } \ell$$

(5.1) $$x_{ijk} \ge 0, \quad \text{all } i\ j\ k \text{ with } i \notin I$$

(5.2) $$\xi_{jk} \ge 0\ , \quad \text{all } j\ k \text{ such that } i\ j\ k \text{ exists for } i \in I$$

(6) $$y_{k\ell} \ge 0\ , \text{ all } k\ell \quad \text{and} \quad (y,z) \in \Omega\ ,$$

where we define

(10) $$D_{I,\ell} \triangleq \sum_{i \in I} D_{i\ell}$$

and where $L(y;b)$ is some linear function of y designed to "compensate" for aggregation error in spite of the arbitrary choice of b.

Notice that the mathematical structure of $\tilde{P}_{I,b}$ is identical to that of P (with the addition to the objective function of a new term linear in y, which seems innocuous enough). $\tilde{P}_{I,b}$ is smaller in that the items of I

have been aggregated together throughout.

The major task at this point is to understand the relationship between P_I and $\tilde{P}_{I,b}$. Our main results in this direction are summarized in the next section.

III. THE RELATIONSHIP BETWEEN PLANNING MODELS P_I and $\tilde{P}_{I,b}$

As it turns out, a natural choice for the L function exists for which a nearly ideal relationship can be established between P_I and $\tilde{P}_{I,b}$. In particular, an à priori bound can be obtained on the difference between their optimal values. Such a bound can be obtained for any choice of b, and in fact furnishes a useful criterion for making this choice.

It will be convenient to refer to the so-called *Range* function, which is defined for any collection $\{\alpha_1,\ldots,\alpha_n\}$ of scalars as

$$\underset{1\le j\le n}{\text{Range}}\ \{\alpha_j\} \triangleq \underset{1\le j\le n}{\text{Max}}\ \{\alpha_j\} - \underset{1\le j\le n}{\text{Min}}\ \{\alpha_j\}\ .$$

The notation $v(\cdot)$ will refer to the optimal value of an optimization problem.

Main Theorem. Assume that the same jk links exist for every item in some subset I. Let b_{jk} be chosen arbitrarily for these links, and take the compensation function L to be

$$L(y;b) = \sum_{k\ell}\left(\sum_{i\varepsilon I} D_{i\ell}\ \underset{j}{\text{Min}}\ \{c_{ijk} - b_{jk}\}\right) y_{k\ell}\ . \tag{11}$$

Then

$$v(\tilde{P}_{I,b}) \le v(P_I) \le v(\tilde{P}_{I,b}) + \epsilon_b\ , \quad \text{where} \tag{12}$$

$$\epsilon_b \triangleq \sum_{\ell}\ \underset{k}{\text{Max}}\ \{\sum_{i\varepsilon I} D_{i\ell}\ \underset{j}{\text{Range}}\ \{c_{ijk}-b_{jk}\}\}. \tag{13}$$

Moreover, a complete ϵ_b-optimal solution of P_I can be obtained from any optimal solution $(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi})$ to $\tilde{P}_{I,b}$ by using $(\tilde{x},\tilde{y},\tilde{z})$ as is and supplementing it by values for the missing x_{ijk} for $i \varepsilon I$ according to the disaggregation formula: for all ijk with $i \varepsilon I$, put

$$\text{(14)} \qquad \tilde{x}_{ijk} = \begin{cases} \dfrac{\sum_{\ell} D_{i\ell}\, \tilde{y}_{k\ell}}{\sum_{\ell} D_{I,\ell}\, \tilde{y}_{k\ell}} \tilde{\xi}_{jk} & \text{if } \tilde{\xi}_{jk} > 0 \\ 0 & \text{if } \tilde{\xi}_{jk} = 0 \end{cases} .$$

The proof is given in Appendix 1, along with a generalization to the case where several subsets of items are aggregated simultaneously. Extensions accomodating suboptimal solutions to $\tilde{P}_{I,b}$ are easy to obtain.

This theorem is a satisfying one in a number of respects. First, it allows for an arbitrary aggregation set I subject to the requirement that the items involved have a common set of transportation links (otherwise feasibility difficulties could be encountered in trying to recover a feasible solution to P_I from one of $\tilde{P}_{I,b}$). Second, it allows an arbitrary choice of b, which accomodates any heuristic rule that may be appealing in a particular situation (e.g., some weighted mean of c_{ijk} over $i \in I$). Third, it selects L in such a manner that the aggregated problem is a relaxation of the original one in a suitable sense, thereby producing an underestimate of the optimal value of the original problem. Fourth, this underestimate has an error that is known à priori to be no larger than a calculable number ϵ_b . Fifth, solving the aggregated problem is guaranteed to furnish a complete ϵ_b-optimal solution to P_I (one can very likely conclude that this solution is ϵ-optimal in P_I for some ϵ smaller than ϵ_b -- just take the difference between the objective function (1) evaluated at the feasible solution and the lower bound $v(\tilde{P}_{I,b})$). And sixth, the explicit formula for ϵ_b has a number of valuable applications. We now expand on this last point.

An important question is how one should select b when a compelling heuristic choice is not available. The formula for ϵ_b furnishes a natural criterion: select b to make ϵ_b as small as possible. Happily, this can be converted to a linear programming problem by using standard tricks (mainly the representation of the maximum of a set of numbers as their least upper bound). Thus <u>the optimal b can always be calculated by linear programming</u>.

The ϵ_b-minimizing choice of b can sometimes be obtained analytically if additional assumptions are imposed. For instance, if the $D_{i\ell}$'s are proportionally the same for i in I at every customer -- i.e., if there exist proportions p_i ($p_i \geq 0$ for $i \epsilon I$ and $\sum_{i \epsilon I} p_i = 1$) such that

(15) $$\frac{D_{i\ell}}{\sum_{i \epsilon I} D_{i\ell}} = p_i \quad \text{for all } i\ell \text{ with } i \epsilon I$$

-- and $p_{i_o} \geq \sum_{\substack{i \epsilon I \\ i \neq i_o}} p_i$ for some $i_o \epsilon I$, then it can be shown that the optimal choice of b is to take $b_{jk} = c_{i_o jk}$ for all jk.

It is of interest to characterize the situations where $\epsilon_b = 0$ is possible. It is shown in Appendix 2 that a necessary and sufficient condition for ϵ_b to equal 0 for some choice of b is that there exist numbers β_{jk} and γ_{ik} such that

(16) $$c_{ijk} = \beta_{jk} + \gamma_{ik} \quad \text{for all } ijk \text{ with } i\epsilon I \text{ and } k \text{ such that it is connected to some } \ell \text{ with } D_{i\ell} > 0 \; .$$

If this condition holds, then $\epsilon_b = 0$ is achieved by taking $b_{jk} = \beta_{jk}$ for all jk (plus any constant depending only on k) with k such that it is connected to some ℓ for which $\sum_{i\epsilon I} D_{i\ell} > 0$. The choice of b_{jk} is arbitrary for any k's left over.

When might (16) hold? An important case occurs when item i has a procurement cost γ_i \$/unit, and all items in I have the same unit in-bound transportation rate when measured on a per mile basis, say t_I \$/unit-mile. If the distance from j to k is d_{jk}, then

(17) $$c_{ijk} = t_I \, d_{jk} + \gamma_i \quad \text{for all } ijk \text{ with } i\epsilon I$$

and (16) clearly holds. This case admits an easy generalization that still leaves ϵ_b=0: t_I can depend on j or k or both, and γ_i can depend on k .

IV. CONCLUSION

We have achieved our goal of providing rigorous guidance to the modeler who wishes to consider aggregating a subset I of items in the procurement portion of a logistics planning model. Assuming that the aggregate constraints (2.2) offer adequate control of the procurement pattern, the modeler can obtain an à priori bound from (13) on the amount of suboptimality tnat will be caused in the model by subsequently collapsing the inbound flows for i in I down to a single transportation-like problem that uses any plausible costs b_{jk} for the aggregated items. It bears emphasis that this bound can be calculated <u>before</u> optimizing the aggregated planning model, perhaps using rough preliminary data, and hence is a useful tool for model design.

The results attained can be used not only to study the effects of aggregation with a predetermined subset I of items, but also to select I itself on the basis of small anticipated aggregation error. This can be done by cluster analysis aimed at finding item subsets for which (16) holds approximately. One way to procede is based on the following observation. Notice that if (16) holds exactly, then summing over j yields

$$\sum_j c_{ijk} = \sum_j \beta_{jk} + ||j||_i \gamma_{ik} \quad \text{for } i\ k\ ,$$

where $||j||_i$ is the number of procurement zones supplying item i. Thus γ_{ik} can be eliminated in (16) using

$$\gamma_{ik} = \frac{\sum_j c_{ijk}}{||j||_i} - \frac{\sum_j \beta_{jk}}{||j||_i}$$

to obtain

(16)' $$c_{ijk} - \sum_{\substack{j\\ \|j\|_i}} c_{ijk} = \beta_{jk} - \sum_{\substack{j\\ \|j\|_i}} \beta_{jk}$$ for all ijk with $i \in I$ and k such that it is connected to some ℓ with $D_{i\ell} > 0$.

Conversely, (16)' implies that (16) holds. Hence (16) and (16)' are equivalent conditions. The obvious clustering approach would be to identify with each item i a linearized vector V^i with typical entry

$$\begin{cases} c_{ijk} - \sum_{\substack{j\\ \|j\|_i}} c_{ijk} & \text{if link ijk exists} \\ \text{a large number } M & \text{otherwise.} \end{cases}$$

The V^i-vectors would then be clustered by some standard technique [1] to discover subsets of i for which the V^i's are nearly identical. These subsets of i would identify items which, if aggregated, would tend to have small aggregation error when an appropriate choice for b is used. In fact, an appropriate choice for b would be a virtual by-product of most standard clustering schemes.

A refinement would be to weight the V^i's or its components according to demand or some measure of the likelihood that a given link would actually be selected by the model for use.

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APPENDIX 1: PROOF OF THE MAIN THEOREM

Let $v(\cdot)$ denote the infimal value of any minimizing optimization problem.

Lemma 1 [6]. Consider the two optimization problems

$$(Q) \quad \text{Minimize } f(w) \text{ subject to } w \in W$$

$$(\tilde{Q}) \quad \text{Minimize } \tilde{f}(w) \text{ subject to } w \in W\ ,$$

where f and $\tilde{f}$ are real-valued functions bounded below on a non-empty set W. (Interpret (Q) as the "true" problem and $(\tilde{Q})$ as the "approximating" problem in the sense that an approximate objective function $\tilde{f}$ replaces f.) Let $\underline{\epsilon}$ and $\overline{\epsilon}$ be scalars (not necessarily nonnegative) satisfying

$$\text{(A1)} \quad -\underline{\epsilon} \leq \tilde{f}(w) - f(w) \leq \overline{\epsilon} \quad \text{for all } w \in W.$$

Then

$$\text{(A2)} \quad -\underline{\epsilon} \leq v(\tilde{Q}) - v(Q) \leq \overline{\epsilon}$$

and any optimal solution $\tilde{w}$ of $(\tilde{Q})$ is necessarily $(\underline{\epsilon} + \overline{\epsilon})$- optimal in (Q).

Lemma 1 will be applied not to P_I in the role of (Q), but rather to an equivalent version of P_I , namely its "projection" [3] onto the variables y,z, and x with $i \notin I$:

$$(P_I)^* \quad \underset{x,y,z}{\text{Minimize}} \quad F(y,z) + \sum_{\substack{i \notin I \\ jk}} c_{ijk}\, x_{ijk} + \varphi_I\,(y)$$

$$\text{subj. to} \quad (2.1), (3.2), (4), (5.1), (6)$$

where we define

$$\text{(A3)} \quad \varphi_I\,(y) \triangleq \text{Infimum} \sum_{\substack{ijk \\ i \in I}} c_{ijk}\, x_{ijk} \quad \text{subj. to (2.2) and}$$

$$\sum_j x_{ijk} = \sum_\ell D_{i\ell}\, y_{k\ell}\ , \quad \text{all } ik \text{ with } i \in I$$

$$x_{ijk} \geq 0\ , \quad \text{all } ijk \text{ with } i \in I$$

Make the identifications

$$w = \text{the variables of } (P_I)^*$$

$$W = \text{the constraints of } (P_I)^*$$

$$f(w) = \text{the objective function of } (P_I)^*$$

$$\tilde{f}(w) = \text{the objective function of } (P_I)^* \text{ with } \varphi_I \text{ replaced by } \tilde{\varphi}_I \text{ ,}$$

where $\tilde{\varphi}_I\ (y)$ is defined as

(A4) $$\tilde{\varphi}_I\ (y) \overset{\Delta}{=} L(y;b) + \operatorname*{Inf.}_{\xi \geq 0} \sum_{jk} b_{jk}\, \xi_{jk} \text{ subj. to (2.2A) and (3.1)}$$

with L as defined in (11) for arbitrary fixed b. The justification for (A4) is provided by

Lemma 2. Assume that the same jk links exist for every item in the subset I. Then

(A5) $$\tilde{\varphi}_I\ (y) \leq \varphi_I\ (y) \leq \tilde{\varphi}_I\ (y) + \epsilon_b \text{ , all } (y,z) \text{ satisfying (4) and (6),}$$

where ϵ_b is defined as in (13).

Once Lemma 2 is established, conclusion (12) of the Main Theorem is at hand upon applying Lemma 1 using the identifications given above and the obvious facts $v(Q) = v(P_I)^* = v(P_I)$ and $v(\tilde{Q}) = v(\tilde{P}_{I,b})$.

<u>Proof of Lemma 2</u>. Introduce a supplementary nonnegative variable ξ_{jk} into (A3) for each jk link in existence for $i \in I$, along with the supplementary constraints $\xi_{jk} = \sum_{i \in I} x_{ijk}$ and the supplementary terms $b_{jk}\xi_{jk} - b_{jk}\xi_{jk}$ in the objective function. From (2.2) we see that additional redundant

constraints (2.2A) may be added, and from the demand constraints of (A3) we see that (3.1) may be added. Clearly none of this alters the infimal value of (A3). Upon "projection" of the augmented problem onto the ξ-variables, one obtains

$$(A3)^* \qquad \varphi_I(y) = \underset{\xi \geq 0}{\text{Infimum}} \ \sum_{jk} b_{jk}\xi_{jk} + R(\xi,y) \text{ subj. to } (2.2A),(3.1)$$

where the remainder term is defined as

$$R(\xi,y) \triangleq \text{Infimum} \sum_{\substack{ijk \\ i\varepsilon I}} (c_{ijk} - b_{jk})\, x_{ijk}$$

subj. to

$$\sum_j x_{ijk} = \sum_\ell D_{i\ell} y_{k\ell}, \text{ all } ik \text{ with } i \ \varepsilon \ I$$

$$\sum_{i\varepsilon I} x_{ijk} = \xi_{jk}, \text{ all } jk$$

$$x_{ijk} \geq 0, \text{ all } ijk \text{ with } i \ \varepsilon \ I \ .$$

It is easy to verify that

$$\underline{R}(y) \leq R(\xi,y) \leq \overline{R}(y) \text{ for all } (y,z) \text{ satisfying (4) and (6)}$$
$$\text{and } \xi \text{ satisfying (2.2A) and (3.1),}$$

where

$$\underline{R}(y) \triangleq \sum_{k\ell} \left(\sum_{i\varepsilon I} D_{i\ell} \underset{j}{\text{Min}} \{c_{ijk} - b_{jk}\} \right) y_{k\ell} \equiv L(y;b) \text{ as defined in (11)}$$

$$\overline{R}(y) \triangleq \sum_{k\ell} \left(\sum_{i\varepsilon I} D_{i\ell} \underset{j}{\text{Max}} \{c_{ijk} - b_{jk}\} \right) y_{k\ell}.$$

Since $\overline{R}(y) - \underline{R}(y)$ clearly is no larger than

$$\sum_\ell \underset{k}{\text{Max}} \left\{ \sum_{i\varepsilon I} D_{i\ell} \left[\underset{j}{\text{Max}} \{c_{ijk} - b_{jk}\} - \underset{j}{\text{Min}} \{c_{ijk} - b_{jk}\} \right] \right\}$$

$$= \sum_{\ell} \max_{k} \left\{ \sum_{i \in I} D_{i\ell} \operatorname*{Range}_{j} \{c_{ijk} - b_{jk}\} \right\} \equiv \epsilon_b \text{ as defined in (13)}$$

for any $y \geqq 0$ satisfying (4), we have

(A 6) $L(y;b) \leq R(\xi,y) \leq L(y;b) + \epsilon_b$ for all (y,z) satisfying (4) and (6) and ξ satisfying (2.2A) and (3.1).

The desired conclusion (A 5) now follows easily from (A.3)* and (A6). This completes the proof of Lemma 2.

Finally we come to the second conclusion of the Main Theorem. Let $(\tilde{x}^-, \tilde{y}, \tilde{z}, \tilde{\xi})$ be any optimal solution to $\tilde{P}_{I,b}$ and generate $\tilde{x}^+_{ijk}$ for $i \in I$ according to

$$\tilde{x}^+_{ijk} = \frac{\sum_{\ell} D_{i\ell} \tilde{y}_{k\ell}}{\sum_{\ell} D_{I,\ell} \tilde{y}_{k\ell}} \tilde{\xi}_{jk}, \quad \text{all } ijk \text{ with } i \in I.$$

This "any feasible disaggregation of $\tilde{\xi}$ " construction is possible because of the assumption that the same jk links exist for all $i \in I$. We must show that $(\tilde{x}^-, \tilde{x}^+, \tilde{y}, \tilde{z})$ is feasible and ϵ_b-optimal in P_I . The verification of feasibility is straightforward. To verify ϵ_b-optimality we need to show

$$\sum_{\substack{ijk \\ i \notin I}} c_{ijk} \tilde{x}^-_{ijk} + \sum_{\substack{ijk \\ i \in I}} c_{ijk} \tilde{x}^+_{ijk} + F(\tilde{y},\tilde{z}) \leq v(P_I) + \epsilon_b .$$

This is an obvious consequence of (12) and

$$v(\tilde{P}_{I,b}) \leq \sum_{ijk} c_{ijk} \tilde{x}_{ijk} + F(\tilde{y},\tilde{z}) \leq v(\tilde{P}_{I,b}) + \epsilon_b .$$

This last result, in turn, is a simple consequence of these two facts:

$$\sum_{\substack{ijk \\ i\notin I}} c_{ijk}\, \bar{\tilde{x}}_{ijk} + \sum_{jk} b_{jk}\tilde{\xi}_{jk} + F(\tilde{y},\tilde{z}) + L(\tilde{y};b) = v(\tilde{P}_{I,b}),$$

which holds by the definition of $(\bar{\tilde{x}}, \tilde{y}, \tilde{z}, \tilde{\xi})$, and

$$L(\tilde{y};b) \leq \sum_{\substack{ijk \\ i\varepsilon I}} (c_{ijk}-b_{jk})\tilde{x}^{+}_{ijk} \leq L(\tilde{y};b) + \epsilon_b \quad ,$$

which can be simplified to

$$L(\tilde{y};b) \leq \sum_{\substack{ijk \\ i\varepsilon I}} c_{ijk}\, \tilde{x}^{+}_{ijk} - \sum_{jk} b_{jk}\, \tilde{\xi}_{jk} \leq L(\tilde{y};b) + \epsilon_b \; .$$

This completes the proof of the Main Theorem.

Remark . It is a straightforward matter to generalize the Main Theorem to cover the case where several disjoint subsets of items are to be aggregated, say $I^1,\ldots,I^H$. The analogs of P_I and $\tilde{P}_{I,b}$ should be obvious. Assume for $h = 1,\ldots,H$ that the same jk links exist for every item in subset I^h and choose b^h_{jk} arbitrarily for these links. Define

$$L^h\,(y;b^h) \triangleq \sum_{k\ell}\left(\sum_{i\varepsilon I^h} D_{i\ell}\, \underset{j}{\mathrm{Min}}\, \{c_{ijk} - b^h_{jk}\}\right) y_{k\ell} \; .$$

Then

$$v(\text{analog of } \tilde{P}_{I,b}) \leq v(\text{analog of } P_I) \leq v(\text{analog of } \tilde{P}_{I,b}) + \epsilon^H_b \; ,$$

where

$$\epsilon^H_b \triangleq \sum_{\ell} \underset{k}{\mathrm{Max}} \left\{ \sum_{h=1}^{H} \sum_{i\varepsilon I^h} D_{i\ell}\, \underset{j}{\mathrm{Range}}\, \{c_{ijk} - b^h_{jk}\} \right\},$$

and an ϵ^H_b-optimal solution of the analog of P_I can be constructed in the obvious way. Note that ϵ^H_b is smaller than the tolerance that would be obtained from H successive applications of the original version of the Main Theorem.

APPENDIX 2:

NECESSARY AND SUFFICIENT CONDITIONS FOR ZERO AGGREGATION ERROR

Proposition $\epsilon_b = 0$ in expression (13) if and only if there exist numbers γ_{ik} such that

$$c_{ijk} = b_{jk} + \gamma_{ik} \qquad \text{for all } ijk \text{ with } i \in I \text{ and } k \text{ such that it is connected to some } \ell \text{ with } D_{i\ell} > 0 .$$

Proof. It is easy to see that $\epsilon_b = 0$ if and only if

$$D_{i\ell} \operatorname*{Range}_{j} \{c_{ijk} - b_{jk}\} = 0 \qquad \text{for all possible } ik\ell \text{ with } i \in I$$

(for $ik\ell$ to be possible, k must be connected to ℓ and ijk must exist for some j)

which, by the nonnegativity of $D_{i\ell}$ and of the range function, holds if and only if

(A7) $$\operatorname*{Range}_{j} \{c_{ijk} - b_{jk}\} = 0 \qquad \text{for all possible } ik \text{ with } i \in I \text{ and } k \text{ such that it is connected to some } \ell \text{ with } D_{i\ell} > 0 .$$

Now the range function has the property that it vanishes if and only if all of its arguments are identical, and so (A7) holds if and only if numbers γ_{ik} exist such that

$$c_{ijk} - b_{jk} = \gamma_{ik} \qquad \text{for all } ijk \text{ with } i \in I \text{ and } k \text{ such that it is connected to some } \ell \text{ with } D_{i\ell} > 0.$$